How to find a hidden clique

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Semidefinite Optimization
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Agenda

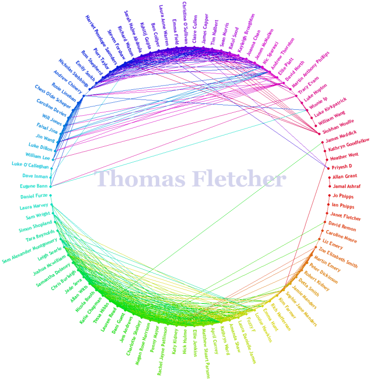
- Present a convex relaxation for the maximum clique problem based on nuclear norm relaxation for rank minimization.
- Extend this relaxation to one for the densest k-subgraph problem.
- Phase transitions: tradeoff between on size of cliques and density ensuring recovery from the relaxations.
- Experimental results and open problems.
- Joint work with Stephen Vavasis, University of Waterloo.

Graph clustering

- Similarity Graph: represent data set as a graph
  - items = nodes
  - edges indicate similarity
- Cluster the data set by dividing the graph into dense subgraphs.
- Dense = large average degree

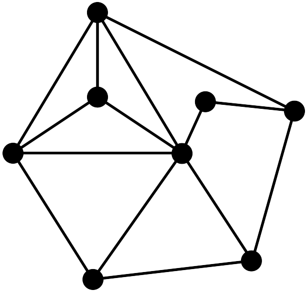
Example: Communities in Social Networks

- Nodes = users
- Edges = “friendship”.
- Densely connected groups = communities



Cliques of a graph

- Given graph $G = (V, E)$, a clique of G is a pairwise adjacent subset of V.
- $C \subseteq V$ is a clique of G if $uv \in E$ for all $u, v \in C$.
- The subgraph $G(C)$ induced by C is complete.



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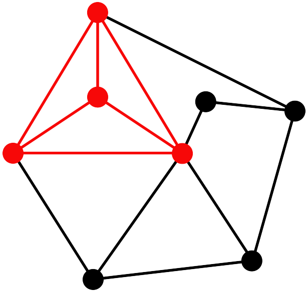


The Maximum Clique problem

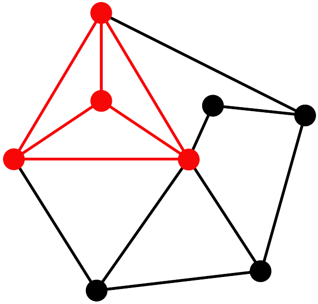
- Optimization version: Find the size of the largest clique of G; size of the largest clique is the clique number $\omega(G)$.
- Decision version: Given graph G, integer k: does G contain a clique of cardinality at least k?
- Complexity: NP-complete, NP-hard to approximate $\omega(G)$ within a ratio of $N^{1-\epsilon}$ for any $\epsilon > 0$, ($N = |V|$)
- Many applications: communication, power, and social networks, mathematical biology, cryptography.

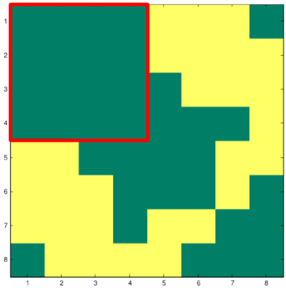
Matrix representation of cliques

- Characteristic vector of C: vector $\mathbf{v} \in \{0,1\}^V$ with

$$v_i = 1 \text{ if } i \in C \quad \text{and} \quad v_i = 0 \text{ otherwise}$$

- If C is a clique with characteristic vector $\mathbf{v}$, let $X = \mathbf{v}\mathbf{v}^T$.
- Nonzero entries of X form a $|C| \times |C|$ all-ones block in $A_G + I$.





Clique as rank minimization

- G has a k-clique if and only if there exists rank-one symmetric binary matrix X such that

$$\sum \sum X_{ij} = k^2$$
$$X_{ij} = 0 \quad \forall\, ij \notin E,\ i \neq j.$$

- Clique is equivalent to the rank minimization problem:

$$\min_{\substack{X \in \{0,1\}^{V \times V} \\ X \in \Sigma^V}} \left\{ \text{rank}(X) : \mathbf{e}^T X \mathbf{e} = k^2, X_{ij} = 0 \text{ if } (i,j) \in \tilde{E} \right\}$$

where $\tilde{E} = V \times V - \left\{ E \cup \{(u,u) : u \in V\} \right\}$.

Nuclear norm relaxation of rank

- Affine rank minimization problem: find matrix with minimum rank satisfying linear constraints:

$$\min\{\mathrm{rank}(X) : \mathcal{A}(X) = \mathbf{b}\}.$$

  Well-known to be NP-hard.

- Relax $\mathrm{rank}(X)$ with nuclear norm $\|X\|_*$:

$$\mathrm{rank}(X) = \|\sigma(X)\|_0, \quad \|X\|_* = \|\sigma(X)\|_1.$$

- If $\mathcal{A}$ is "nice" then the minimum nuclear norm solution is the minimum rank solution.

- Can be written as an SDP.

Nuclear norm relaxation of Clique

- We have the rank minimization problem:

$$\min_{\substack{X \in \{0,1\}^{V \times V} \\ X \in \Sigma^V}} \left\{ \operatorname{rank}(X) : \mathbf{e}^T X \mathbf{e} = k^2,\ X_{ij} = 0 \text{ if } (i,j) \in \tilde{E} \right\}$$

- Relax rank with the nuclear norm and ignore the binary and symmetry constraints:

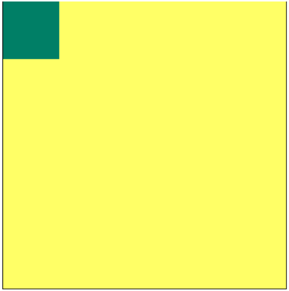
$$\min \left\{ \|X\|_* : \mathbf{e}^T X \mathbf{e} = k^2,\ X_{ij} = 0 \text{ if } (i,j) \in \tilde{E} \right\} \quad (\textbf{NNR})$$

- When does the solution of the relaxation coincide with that of the rank minimization problem?

The planted case

Construction:

- Add edges between nodes in vertex set V^* of size k.



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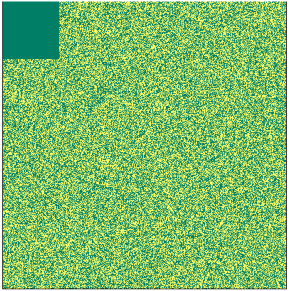
Construction:

- Add edges between nodes in vertex set V^* of size k.
- Each remaining edge is added to E independently with fixed probability p.



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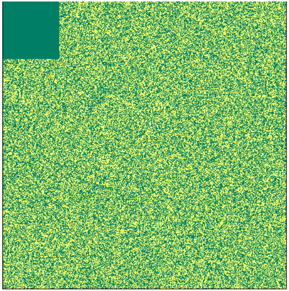
Construction:

- Add edges between nodes in vertex set V^* of size k.
- Each remaining edge is added to E independently with fixed probability p.
- V^* is a clique of G (called a planted or hidden clique).



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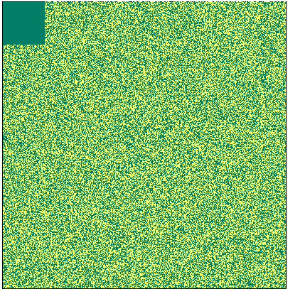

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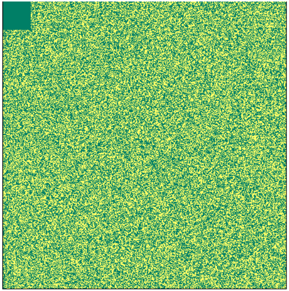

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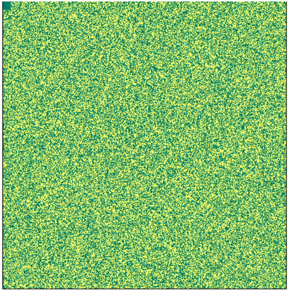


Recovery guarantee in the planted case

Theorem (Ames-Vavasis 2009)

There exists scalar $c > 0$ *(depending only on p) such that if*

$$k \geq c\sqrt{N}$$

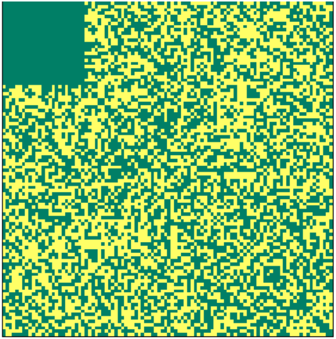
then

- V^* *is the unique maximum clique of* G*, and*
- $X^* = \mathbf{v}\mathbf{v}^T$ *is the unique optimal solution of* $(\mathbf{NNR})$

with high probability.

What happens if edges are deleted?

- Guarantee does not tolerate edge deletion noise.
- Suppose edge uv is deleted for some $u, v \in V^*$.
- Then V^* is not a clique and X^* is not feasible for (**NNR**).



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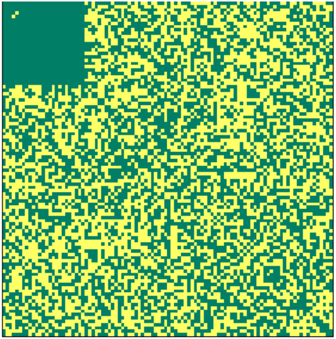


What if p varies with N?

- Guarantee assumes noise probability p is fixed.
- Not a realistic model of many real-world networks.
- Want a bound that allows p to grow/shrink with N.

The densest k-subgraph problem

- Want a dense subgraph of size k, not necessarily a clique.
- Densest k-subgraph problem (DKS): find subgraph $H \subseteq G$ on k nodes with maximum density:

$$d(H) = \frac{|E(H)|}{|V(H)|} = \frac{|E(H)|}{k}.$$

- NP-hard: proof is by reduction to Clique; hard to approximate.
- Maximizing $d(H)$ = maximizing $|E(H)|$ over all k-node subgraphs.

Duality of density and number of missing edges

- Let $V^* \subseteq V$ be a k-subset with characteristic vector $\mathbf{v}$.
- Introduce a correction Y for entries of $X = \mathbf{v}\mathbf{v}^T$ that should be 0:

$$Y_{ij} = \begin{cases} -X_{ij}, & \text{if } ij \in \tilde{E} \\ 0, & \text{otherwise.} \end{cases}$$

- If V^* is almost a clique then $G(V^*)$ should be very dense and Y should be very sparse.
- Cardinality of Y acts as a dual of density of $G(V^*)$:

$$|E(G(V^*))| = \binom{k}{2} - \frac{\|Y^*\|_0}{2}$$

Relaxation as Principal Component Pursuit

- Can relax (**DKS**) as

$$\begin{array}{rl} \min & \|X\|_* + \gamma\|Y\|_1 \\ \text{st} & \mathbf{e}^T X \mathbf{e} = k^2 \\ & X_{ij} + Y_{ij} = 0 \text{ if } ij \in \tilde{E} \\ & X \in [0,1]^{V \times V} \end{array} \qquad (\textbf{DKSR})$$

  where γ is a regularization parameter.

- Relax $\|Y\|_0$ using the ℓ_1-norm $\|Y\|_1$, $\mathrm{rank}(X)$ with the nuclear norm $\|X\|_*$.

Robust PCA

- Robust PCA: decompose matrix M as

$$M = L + S$$

  where L has low-rank, S is sparse.

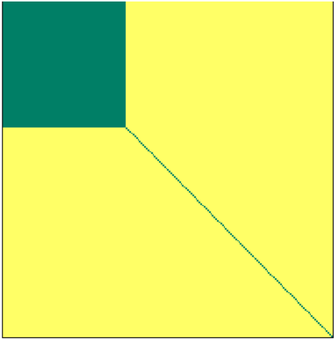
- Chandrasekaran et al 2009, Candès et al 2009, Doan and Vavasis 2010, Chen et al. 2011, Oymak and Hassibi 2011, Jalali et al 2011, Chen et al 2012:

$$\begin{aligned} &\arg\min \left\{ \operatorname{rank}(L) + \gamma \|S\|_0 : P_\Omega(L+S) = P_\Omega(M) \right\} \\ &= \arg\min \left\{ \|L\|_* + \gamma \|S\|_1 : P_\Omega(L+S) = P_\Omega(M) \right\}. \end{aligned}$$

  under certain assumptions on the support of S, column/row space of L, choice of γ, and the set of observed entries Ω.

Planted case

- Start with set of N nodes V.
- Add all edges between nodes in $V^* \subseteq V$.
- Add noise:
  - Add potential edges with probability $p(N)$
  - Delete some edges in $V^* \times V^*$ with probability $q(N)$



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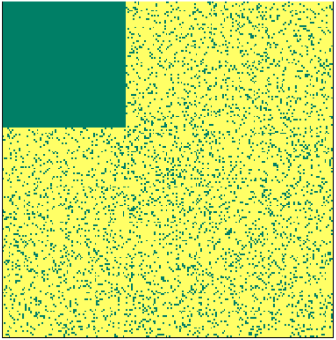

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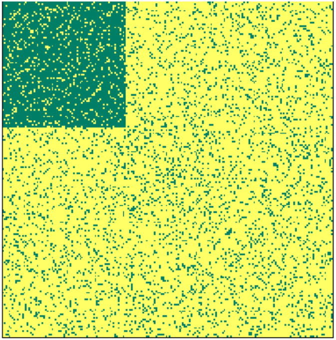


Recovery guarantee

Theorem (Ames 2013)

Under certain technical assumptions on p, q, k, N, there exist absolute constants $c_1, c_2, c_3 > 0$ such that if

$$(1-p-q)(1-p)k \geq c_1 \max\left\{\sqrt{p}, 1/\sqrt{(1-p)k}\right\} \cdot \sqrt{N} \log N$$

and

$$\gamma \in \left(\frac{c_2}{(1-p-q)k}, \frac{c_3}{(1-p-q)k}\right)$$

then

- $G(V^*)$ *is the unique densest* k*-subgraph of* G*, and*
- $X^* = \mathbf{v}\mathbf{v}^T$ *is the unique optimal solution of* (**DKSR**)

with high probability.

Translating the Recovery guarantee

- If p, q are fixed:
  - Get the same bound as before $k = \Omega(\sqrt{N} \log N)$ (with added log term).

- If $p, q \to 0$ as $N \to \infty$ we can find smaller planted cliques.
  - e.g., if $p, q = \Theta(\log k / k)$ can find the planted clique if

$$k = \Omega(N^{1/3} \log N)$$

Proof Idea

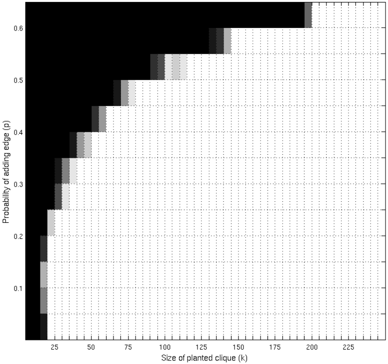
- Apply KKT conditions and SDP duality to derive conditions ensuring optimality and uniqueness of X^*.
- Solve for a choice of multipliers corresponding to X^*.
- Use random matrix theory to show optimality and uniqueness conditions are satisfied (w.h.p.).

Numerical results

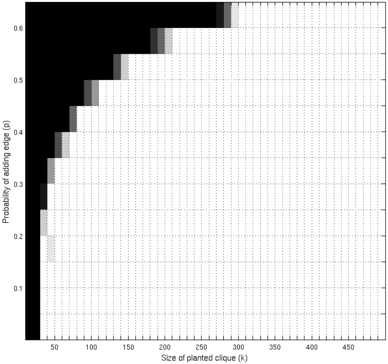
- Randomly generated N-node graphs containing planted dense k-subgraphs.
- Probability of deletion $q = 0.25$ used in each trial, p varied.
- 10 trials for each (p, k) pair.
- Solved **(DKSR)** using ADMM:
    - Move constraints to objective and split the variables so the new objective is separable.
    - Alternately minimize the augmented Lagrangian with respect to each decision variable.
- Compared obtained solution with the planted solution.

Numerical results

$N = 250$



$N = 500$



Conclusions

- New convex relaxations for the Clique and Densest k-subgraph problems.
- Theoretical guarantees for exact recovery in random case.
- Analogous recovery guarantees for deterministic construction.
- Identical results for bipartite graphs.

Conclusions

- Open problems:
  - How to efficiently solve the relaxations?
  - Different sparsity inducing penalties?
  - Many hidden cliques?
  - Are the random bounds tight? Can we relax $\Omega(N^{1/2})$ to $\Omega(N^{1/2-\epsilon})$ for fixed p?
- References:

  - B. Ames and S. Vavasis. Nuclear norm minimization for the planted clique and biclique problems. *Mathematical Programming*, 129(1):121, 2011.
  - B. Ames. Robust convex relaxation for the planted clique and densest k-subgraph problem. 2013. arxiv.org/abs/1305.4891