Convex Relaxation for the Clique and Clustering Problems

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MB2: Semidefinite and Convex Relaxations for Discrete Optimization Problems

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- Data mining/Information retrieval: want to find a pattern hidden in an apparently unstructured set of data
- These problems are intractible in general.
- What if the data consists of the desired structure hidden by random noise?
- e.g. Clustering: if the data is actually clustered, can we identify the clusters efficiently?

Rank minimization

- Affine rank minimization problem **(RMP)** : find matrix with minimum rank satisfying linear constraints:

$$\min\{rank(X) : \mathcal{A}(X) = \mathbf{b}\}$$

(here $\mathcal{A} : \mathbf{R}^{m\times n} \to \mathbf{R}^p$ linear, $\mathbf{b} \in \mathbf{R}^p$).

- NP-hard but can be solved in the case that $\mathcal{A}$ is well-structured by relaxing to convex programming.
- Replace $rank(X)$ with nuclear norm $\|X\|_* = \sigma_1(X) + \cdots + \sigma_m(X)$. **(NNM)**
- Fazel (2002), Recht-Fazel-Parrilo (2007), Candés-Recht (2008).

Cliques and rank-one matrices

- Given a graph $G = (V, E)$, a set of nodes $C \subseteq V$ is clique of G if $ij \in E$ for all $i, j \in C$.
  - C induces a complete subgraph of G.
- Relation to rank-min: every clique C (with char vector $\mathbf{v}$) defines rank-one matrix $\mathbf{v}\mathbf{v}^T$.
  - Block in adjacency matrix of G plus extra 1s on diagonal.
- [A-V 09] if input graph contains sufficiently large clique ($\Omega(\sqrt{N})$), can solve the **maximum clique problem** in polynomial time by formulating as **RMP** and relaxing to **NNM**.

Cliques and Clusters

- Clustering: Given set of data, group them into clusters such that items in each cluster are similar each other and items not in the same cluster are dissimilar.
- Can model data as graph $G = (V, E)$:
  - V is the set of items in the data.
  - Similarity is indicated by E: either by adjacency ($ij \in E$ iff i, j are similar) or by edge-weights.
- If we find optimal partitioning of the graph into cliques, then have an optimal clustering of the data.

The k-disjoint-clique problem

- Given graph $G = (V, E)$ with $|V| = N$ and integer $k \in [1, N]$, a k-disjoint-clique-subgraph is a subgraph of G composed of k disjoint cliques.
- Maximum node k-disjoint-clique problem **(KDC)**: find a k-disjoint-clique subgraph containing the maximum number of nodes of G.
- Mean weight k-disjoint-clique problem **(WKDC)**: Given complete graph K_N and edge-weights W, find a k-disjoint-clique subgraph of K_N that maximizes the sum of average weights covered by each clique.

What if we want to find *many* cliques?

- Can't formulate k-disjoint-clique as rank minimization.
- k may be very large, don't want to recover a low-rank matrix.
- Solution: Relax rank constraints with combination of nuclear norm constraint and semidefinite constraint.

Relaxation as SDP

- Max node k-disjoint-clique can be relaxed as

$$
(*)\qquad
\begin{aligned}
\max \quad & \sum\sum X_{ij} \\
\text{s.t.} \quad & X\mathbf{e} \le \mathbf{e}, \\
& X_{ij} = 0, \qquad \forall (i,j) \notin E, i \neq j \\
& tr(X) = k, \\
& X \succeq 0
\end{aligned}
$$

- $X\mathbf{e} \le \mathbf{e} \Rightarrow \|X\|_* \le rank(X)$ for all feasible X.
- $X \succeq 0 \Rightarrow \|X\|_* = tr(X)$ for all feasible X.

The planted case

- Consider graphs constructed as follows:
- Start with disjoint cliques $C_1, \ldots, C_k$ of size $r_1, \ldots, r_k$.
- Noise: add set C_{k+1} containing r_{k+1} additional nodes and additional edges either deterministically by an adversary or at random independently with fixed prob p.
- $C_1, \ldots, C_k$ induce a feasible solution of $(*)$:

$$X^* = \sum_{i=1}^{k} \frac{1}{r_i} \mathbf{v}_i \mathbf{v}_i^T$$

where $\mathbf{v}_i \in \mathbf{R}^V$ is the characteristic vector of C_i.

Results

- if not too much extra noise is added then, we can recover X^* (and hence $\{C_1, \ldots, C_k\}$) by solving $(*)$.
- For both formulations, can add at most $r_{k+1} = O(\hat{r}^2)$ extra nodes, where $\hat{r} = \min_{i=1,\ldots,k} r_i$.

- Adv case: Can add up to $O(\hat{r}^2)$ additional edges, provided there is at most $O(\min\{r_q, r_{cl(v)}\})$ edges from v to C_q.
- Random case: "too many" noise edges quantified by number of cliques k and the discrepancy between their sizes: need

$$\left(\sum_{s=1}^{k} {r_s}^2\right)^{1/2} \left(\sum_{q=1}^{k} \frac{1}{r_q}\right)^{1/2} \leq O(\hat{r}).$$

Results: Random noise

- Max clique $k = 1$: X^* optimal if $r_1 = \Omega(\sqrt{N})$.
- Constant clique size: $r_1 = \cdots = r_k = N^\alpha$. Can find at most $k = O(N^{\alpha/2})$ disjoint planted cliques for $\alpha \in [1/2, 2/3]$.
- Different sized cliques: e.g. 1 large clique of size $O(N^{2/3})$ and O($N^{1/6}$) smaller cliques of size $O(N^{1/2})$.

Weighted case SDP relaxation

- Relax **(WKDC)** as

$$
(w*) \qquad \begin{aligned} \max \quad & tr(WX) \\ \text{s.t.} \quad & X\mathbf{e} \le \mathbf{e}, \\ & X \ge 0, \\ & tr(X) = k, \\ & X \succeq 0 \end{aligned}
$$

Weighted planted case

- Unlike **(KDC)**, planted case does not correspond to any particular structure in input graph.
- Instead, is induced by edge-weight matrix W: entries of W corresponding to the planted k-disjoint-clique subgraph are larger than the rest.
- Consider random symmetric $k+1 \times k+1$ block matrix $W \in \Sigma^N$ with i.i.d. entries in $[0,1]$ in each block such that

$$E[W_{C_q,C_s}] = E[W_{C_s,C_q}] = \begin{cases} \alpha \mathbf{e}\mathbf{e}^T, & \text{if } q = s,\ 1 \le q,s \le k \\ \beta \mathbf{e}\mathbf{e}^T, & \text{otherwise} \end{cases}$$

for $\alpha > 2\beta$ and partitioning $\{C_1, \dots, C_{k+1}\}$ of V.

Results: weighted case

- If not too much noise, can solve **(WKDC)** by solving $(w*)$.
- Can tolerate up to $O(\hat{r})$ additional nodes.
- As before, too much edge noise is quantified by number of cliques k and the discrepancy between their sizes: need

$$\left(\sum_{s=1}^{k+1} r_s \right)^{1/2} \leq O(\hat{r}).$$

Rehnquist Supreme Court

- Data set is the set of U.S. Supreme Court Justices (serving from 1994-95 to 2003-04).
- Assign edge-weights corresponding to fraction of decisions on which Justices agreed:

	St	Br	Gi	So	Oc	Ke	Re	Sc	Th
1 St	1	0.62	0.66	0.63	0.33	0.36	0.25	0.14	0.15
2 Br	0.62	1	0.72	0.71	0.55	0.47	0.43	0.25	0.24
3 Gi	0.66	0.72	1	0.78	0.47	0.49	0.43	0.28	0.26
4 So	0.63	0.71	0.78	1	0.55	0.5	0.44	0.31	0.29
5 Oc	0.33	0.55	0.47	0.55	1	0.67	0.71	0.54	0.54
6 Ke	0.36	0.47	0.49	0.5	0.67	1	0.77	0.58	0.59
7 Re	0.25	0.43	0.43	0.44	0.71	0.77	1	0.66	0.68
8 Sc	0.14	0.25	0.28	0.31	0.54	0.58	0.66	1	0.79
9 Th	0.15	0.24	0.26	0.29	0.54	0.59	0.68	0.79	1

Rehnquist Supreme Court: results

- We solved **WKDC** with $k = 2$ using SeDuMi. We obtained the following partition of the supreme court:

1: "Liberal"	2: "Conservative"
Stevens (St)	O'Connor (Oc)
Breyer (Br)	Kennedy (Ke)
Ginsberg (Gi)	Rehnquist (Re)
Souter (So)	Scalia (Sc)
	Thomas (Th)

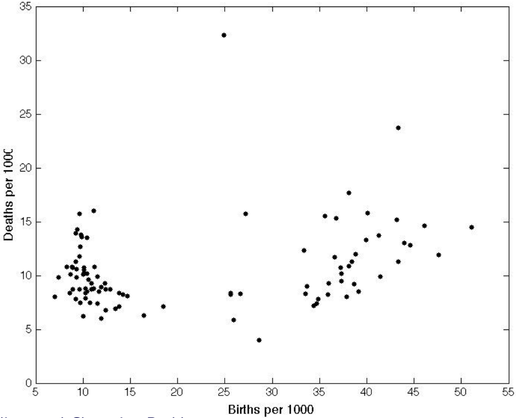
Rehnquist Supreme Court: results

- Algorithm is sensitive to choice of k.
- Solve with $k = 3$:

1: “Most Conservative”	2: “Moderate Conservative”	3: ”Liberal”
Thomas (Th) Scalia (Sc)	O’Connor (Oc) Kennedy (Ke) Rehnquist (Re)	Stevens (St) Breyer (Br) Ginsberg (Gi) Souter (So)

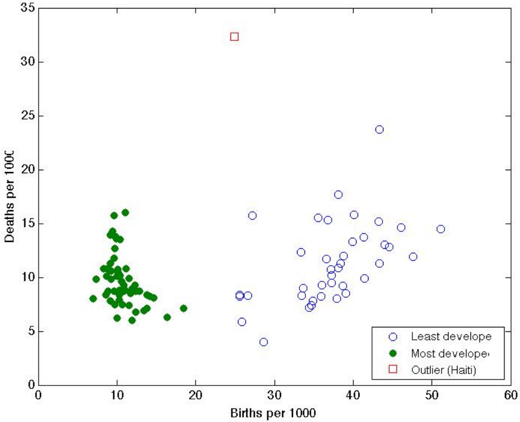
Birth/Death rates

- Source: *2010 Crude Birth & Death, Net Migration, & Growth Rates, Births, Deaths, & Migrants*, U.S. Census Bureau, International Data Base.



Birth/Death rates: results

- Use $W = \exp(-\|\mathbf{v}_i - \mathbf{v}_j\|^2/\sigma^2)$.
- Solve using SDPNAL with $k = 2$:



Conclusions

- The clique and clustering problems be relaxed as SDPs.
- We obtain the exact solution in polynomial time if the data contains the desired hidden structure.
- Future work:
  - extension to other problems: densest subgraph, other graph-partitioning objectives (e.g. normalized cut).
  - unify spectral clustering results: if in planted case is the partitioning given by eigenvectors of W exact?

Thank you!

- B. Ames and S. Vavasis (2010). *Convex optimization for the planted k-disjoint-clique problem* arXiv:1008.2814
- B. Ames and S. Vavasis (2009). *Nuclear norm minimization for the planted clique and biclique problems* arXiv:0901.3348