Finding overlapping communities

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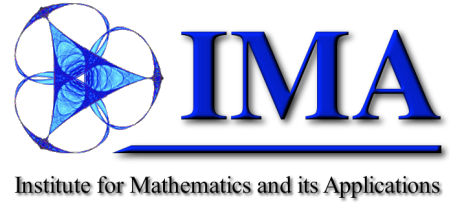


Outline

1. Finding Overlapping Communities in Social Networks. Arora, Ge, Sachdeva, Schoenebeck. 2011

2. What I've been working on (joint with Steve Vavasis and Ting Kei Pong, C&O Dept, University of Waterloo)

3. What's left to do?

Arora et al.

- Traditional clustering techniques exploit the fact that the communities are nonoverlapping when looking for communities.
- Not representative of real-world networks where users can belong to several different communities.
- Present a planted case analysis: Propose a quasipolynomial time algorithm for finding overlapping communities in networks constructed in a particular way.

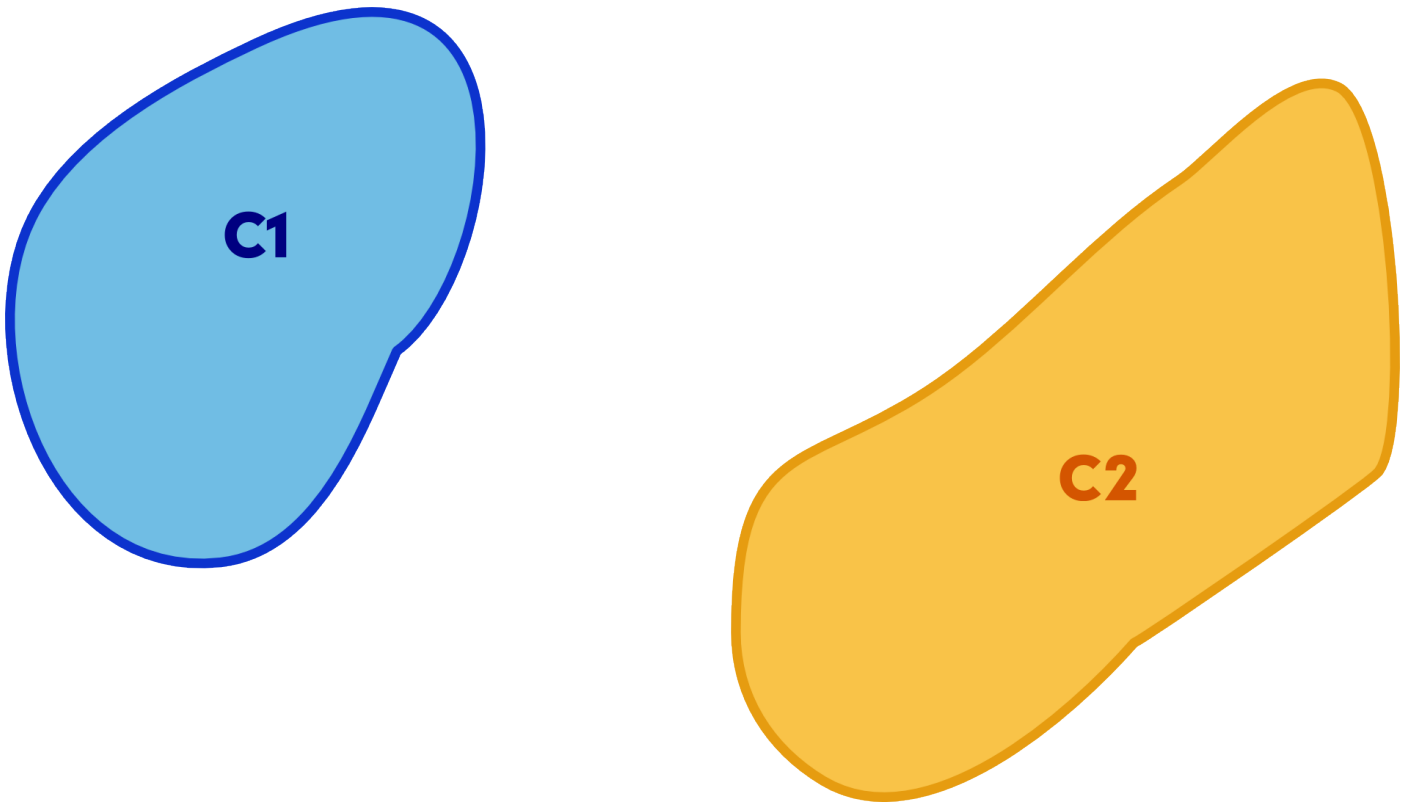
Communities in networks

- Social network may be represented as a graph.
- Nodes correspond to members of the network.
- Edges indicate “friendship”.
- Communities: groups of nodes connected with “lots of edges”.
- Cliques: groups of fully connected nodes (all users are friends).
- Not limited to social networks: internet search, communication/power networks, biological interactions, etc.

The nonoverlapping case

- Most heuristics are clustering based: partition the graph into clusters of similar users.
- Implicitly exploit the fact that the communities are separable (such a partitioning is appropriate):
  - i.e. communities are nonexpanding: densely connected with few edges leaving.
- Graph partitioning is intractable in general.
- In special cases can find nonoverlapping communities:
  - Ng et al. 2002, Ostrovsky et al. 2006, Ames & Vavasis 2010, Rohe et al 2010, Oymak & Hassibi 2011, Jalali et al 2011, Balakrishnan et al. 2011, Ames 2012

Example



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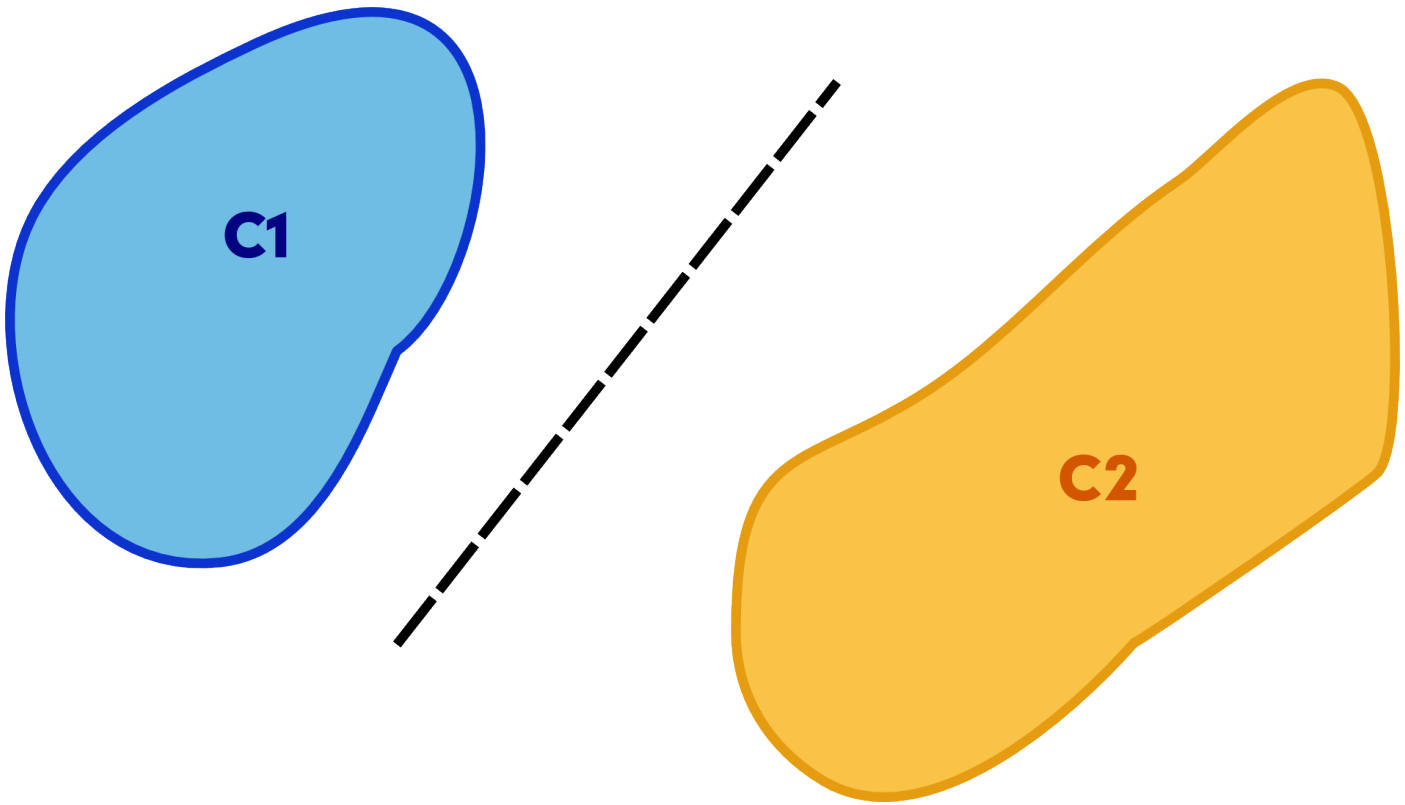

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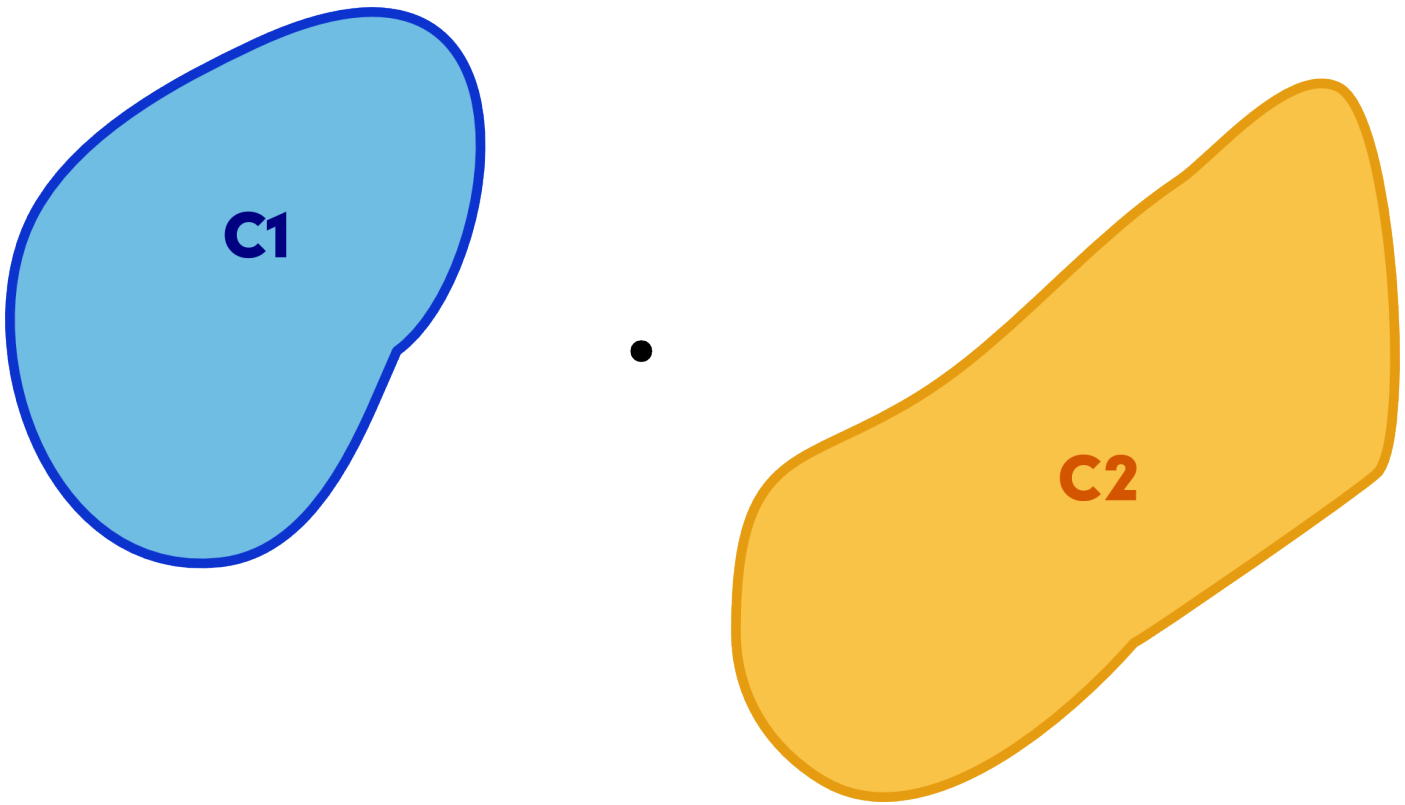

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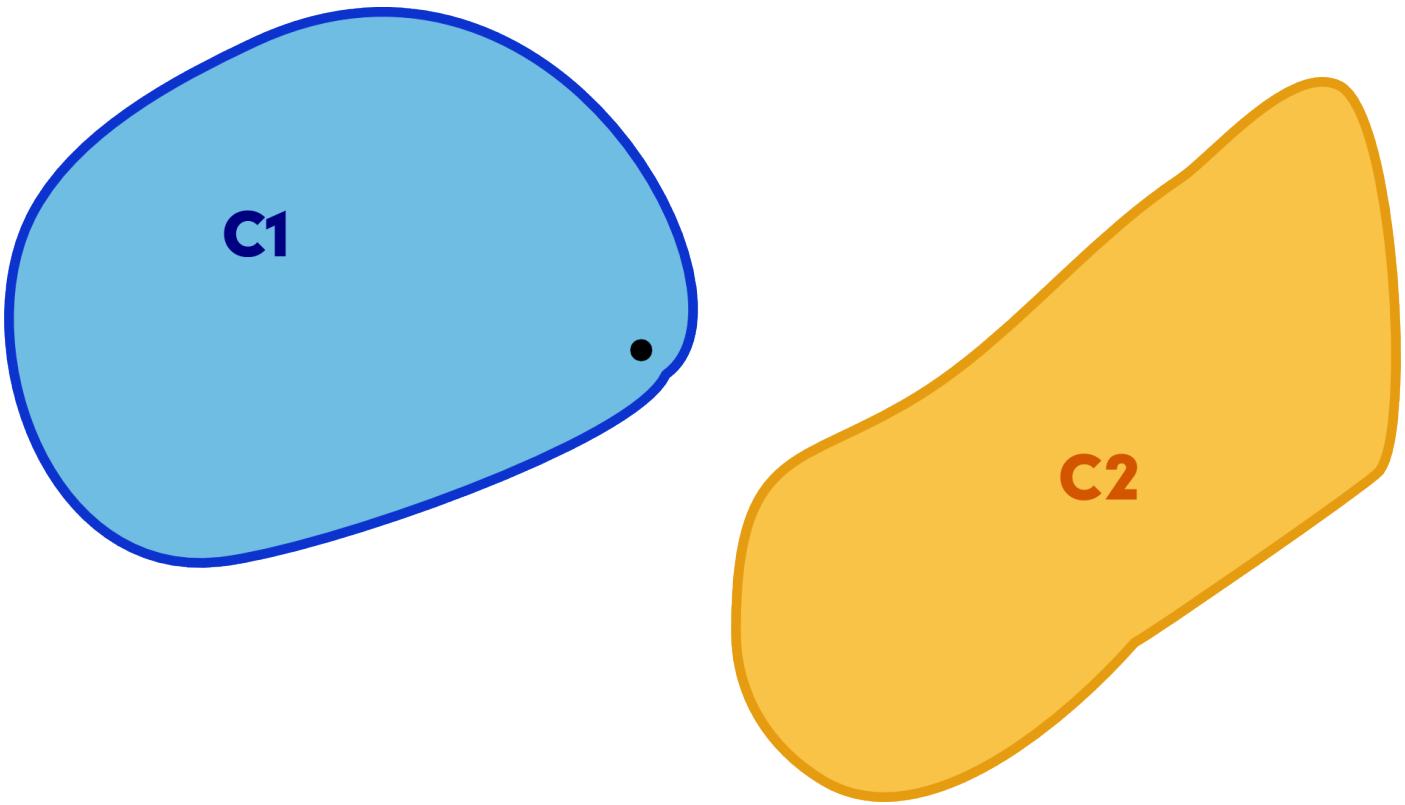

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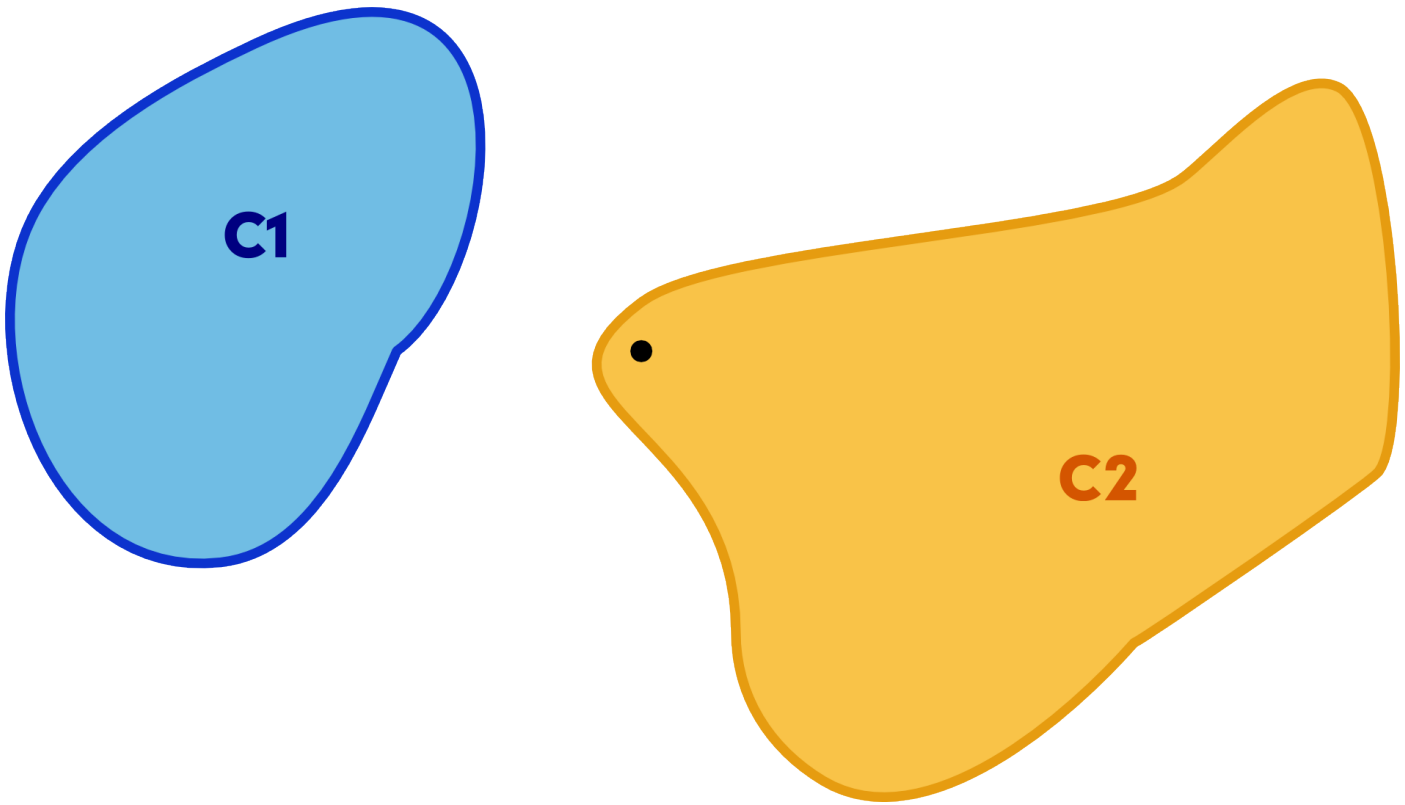


Ego-centric networks

- Model of local behaviour.
- Consist of a single user (ego) and its neighbours.
- Thorougly studied in sociology.
- Individuals typically participate in only a small number of communities.
- Community membership based on homophily: people with shared interests will be friends.
- Have foci (e.g. family, workplace, social activities) around which joint activities are organized.

Notation

- Network is a graph $G = (V, E)$ on n users.
- Each user $u \in V$ belongs to exactly d communities.
- The largest community has cardinality k.

The Expected Degree Model

Assumption (Expected Degree Model)

- *Each node u in community C has affinity $p_u \in [\sqrt{\alpha}, 1]$ for some constant α.*
- *Have edge uv with probability $p_u p_v \in [\alpha, 1]$ if $u, v \in C$.*
- *If u and v are in the overlap of several communities, the probability of uv is the maximum of the $p_u p_v$'s.*

Two more assumptions

Assumption (The Gap Assumption)

A $w \notin C$ is adjacent to at most $(\alpha - \epsilon)|C|$ members of C.

- *i.e. nodes are better attached to the community than nodes outside the community.*
- *Should not be able to add nodes to a community.*

Assumption (Membership Assumption)

For any node, community membership accounts for fraction $\gamma > 0$ of the edges incident with it.

Special case: Cliques of same size

Theorem

- *Suppose that every node has community affinity* $\alpha = 1$ *(every community is a clique).*
- *Suppose further that all communities have size* $|C| = k$*.*
- *Then we can find every community with probability at least* $2/3$ *in time* $O(nk)2^{O(\log^2 d)}$*.*

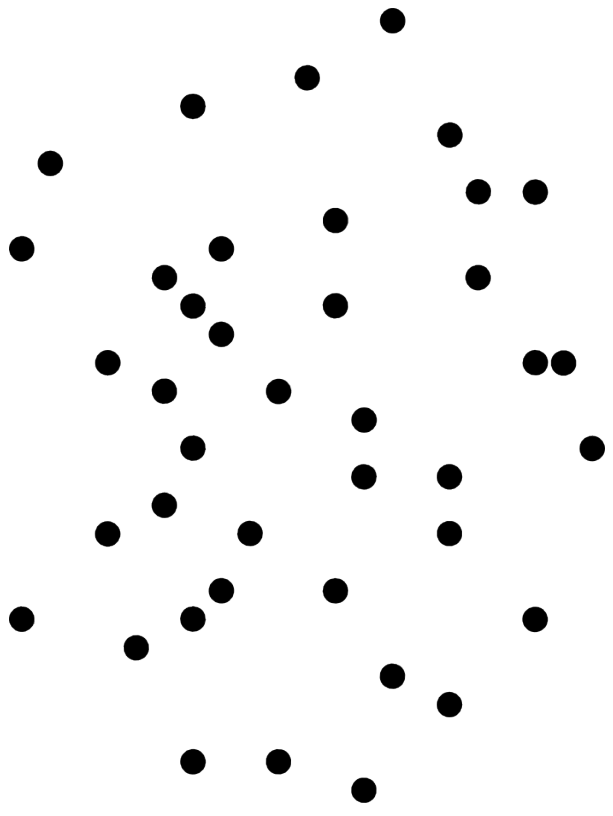
Step 1: Random sampling

Step (Random sampling)

Choose $9n/k$ nodes uniformly at random.

Claim

- *A node is chosen from any community with probability k/n.*
- *Choose ≈ 9 nodes from each community.*



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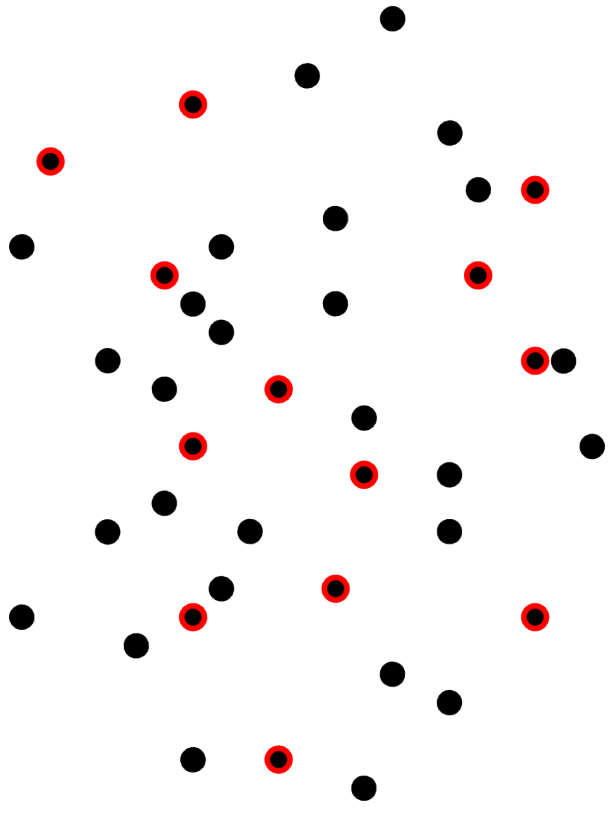
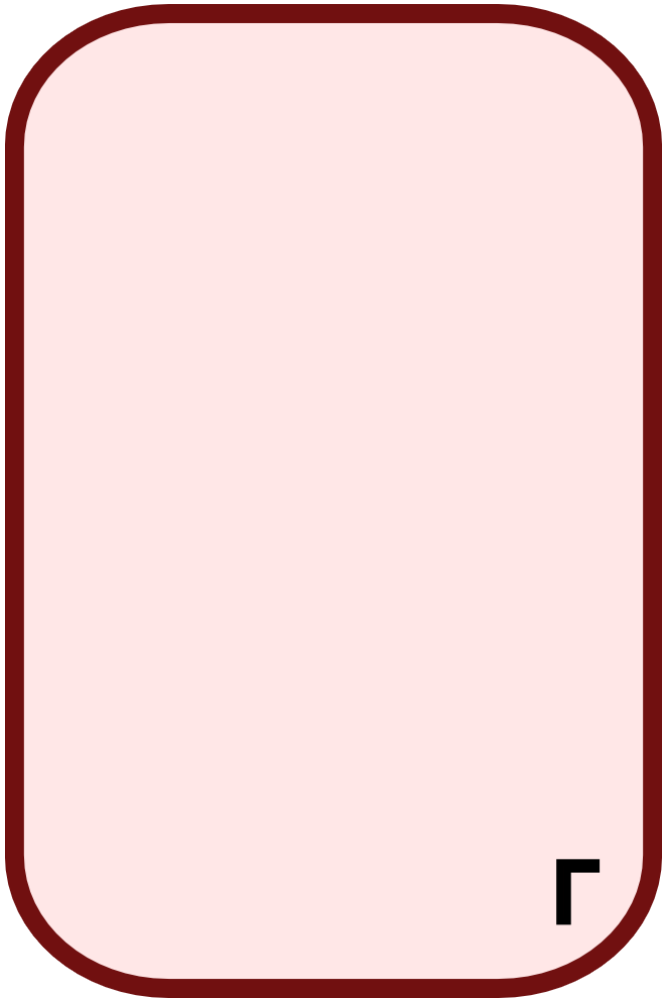


Step 2: Neighbourhood subsampling

Step (Subsampling)

- *Pick sampled node v and find neighbourhood Γ of v.*
- *Randomly sample $S \subseteq \Gamma$ by including each $u \in \Gamma$ independently with fixed probability $p = c/k$ (c scalar depending on d, ϵ, γ).*



Claim

- $|S| \leq 3E[|S|] = O(d \log d)$ *w.h.p.*

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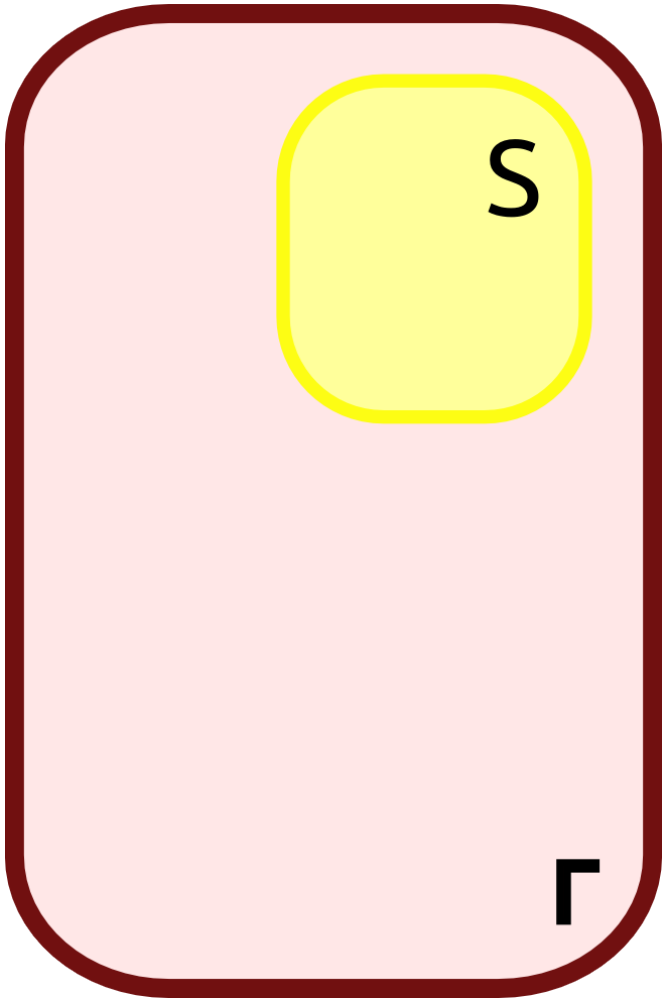

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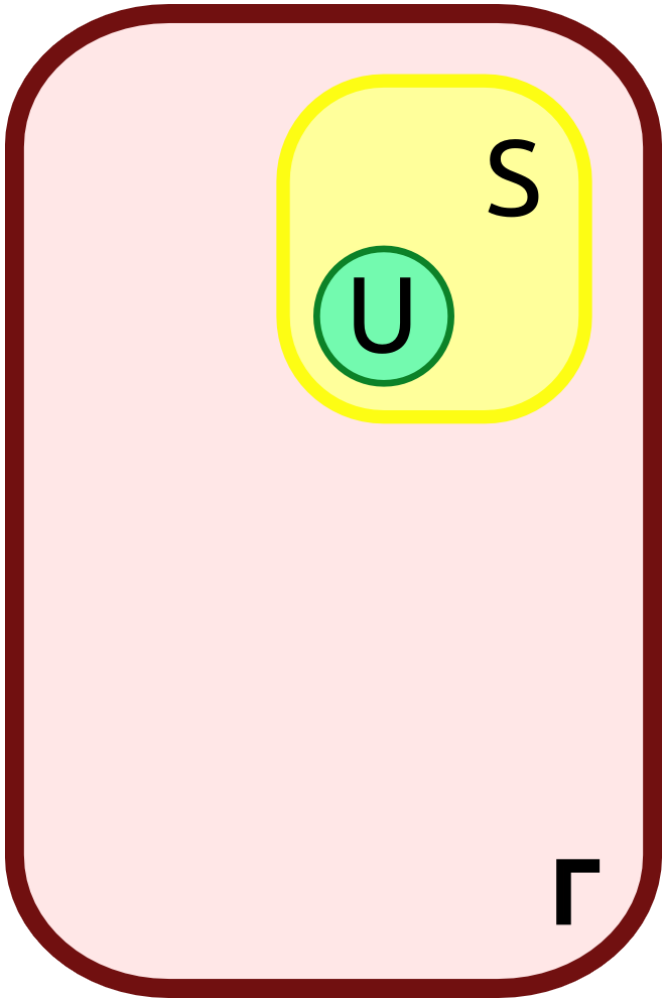
Step 3a: Clique finding

Step (Clique finding)

Find all cliques U of $G(S)$ of size at most $2pk$.

Claim

- *The set of such U includes $S \cap C$ for every community C containing v w.h.p.*



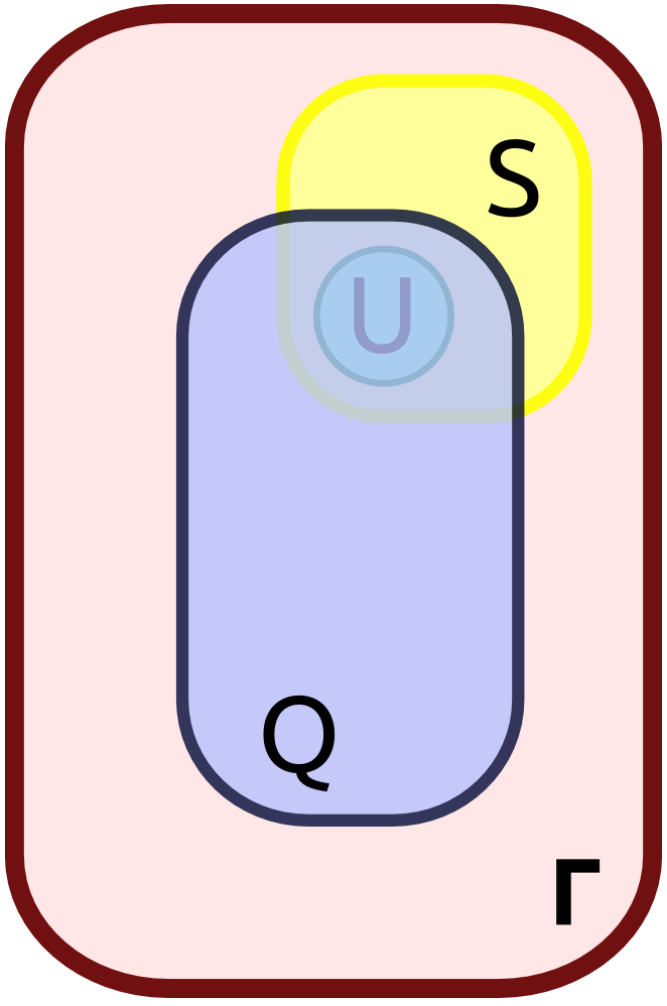
Step 3b: Expand cliques

Step (Expand U)

For all U: find set $Q \subseteq \Gamma$ connected to all nodes of U.

Claim

- *If $U = S \cap C$ then $C \subseteq Q$.*
- $|Q| \leq (1 + \epsilon/4)k$ *w.h.p.*
- $|E(Q)| \leq (1 + \epsilon/12)k$.



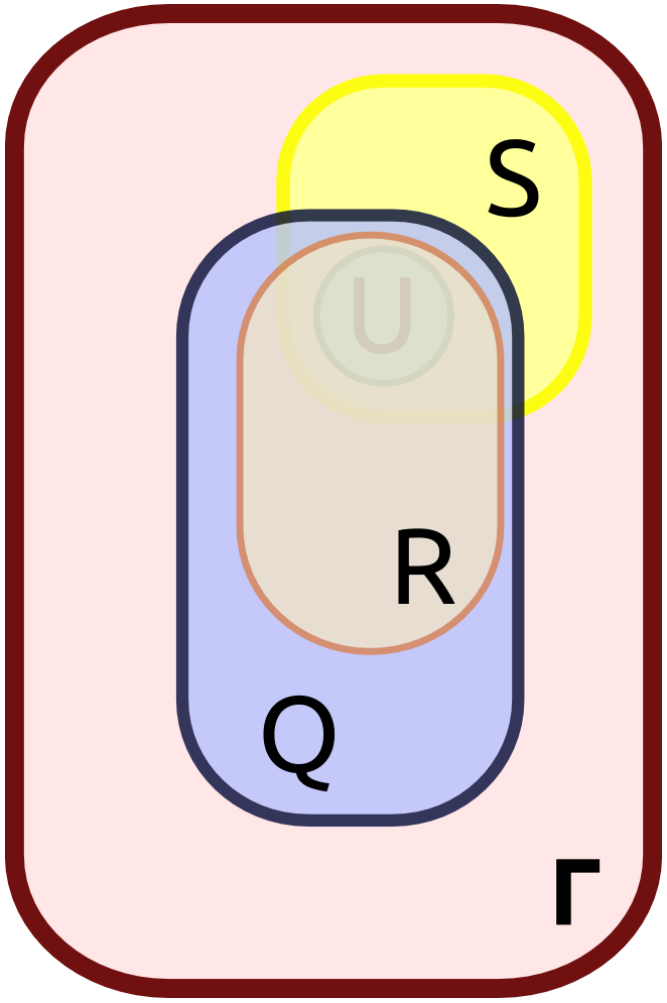
Step 3c: Output communities

Step (Output C)

- *Find set R of nodes with degree at least $(1 - \epsilon/2)|Q|$ in $G(Q)$.*
- *Output $R = C$ if it is a k-clique and satisfies the Gap assumption.*

Claim

- *Nodes in C have degree at least k.*
- *All others $\leq (1 - 3\epsilon/4)k$.*



Complexity of the algorithm

- $pk = O(d \log d)$ and d is small relative to n and k, we can find all cliques of S by exhaustive search quickly.
- Finding the set Q of neighbours of U requires

$$O(2pk|\Gamma|) = O\left(2pk^2 d/\gamma\right)$$

  operations since $|\Gamma| \leq kd/\gamma$.
- Finding R requires checking the degrees of each node in $G(Q)$, which can be done in time $O(k^2)$ since $|Q| = O(k)$.
- Putting everything together, we have a total runtime of at most

$$O(nk/\gamma) \times \text{max number of cliques} = O(nk/\gamma)2^{O(polylog(d))}.$$

The General Case

Theorem

- *Suppose that*
  - *each community satisfies the Expected Degree Model Assumption with* $p_u \geq \sqrt{\alpha_{\min}}$,
  - *the Gap Assumption is satisfied w.r.t* ϵ, *and*
  - *the Membership Assumption is satisfied w.r.t* γ.
- *Then all communities can be recovered in*

$$O\left(n^{C\log(kd/\gamma)/\alpha_{\min}\epsilon^2}\right)$$

time.

The General Case: Intuition

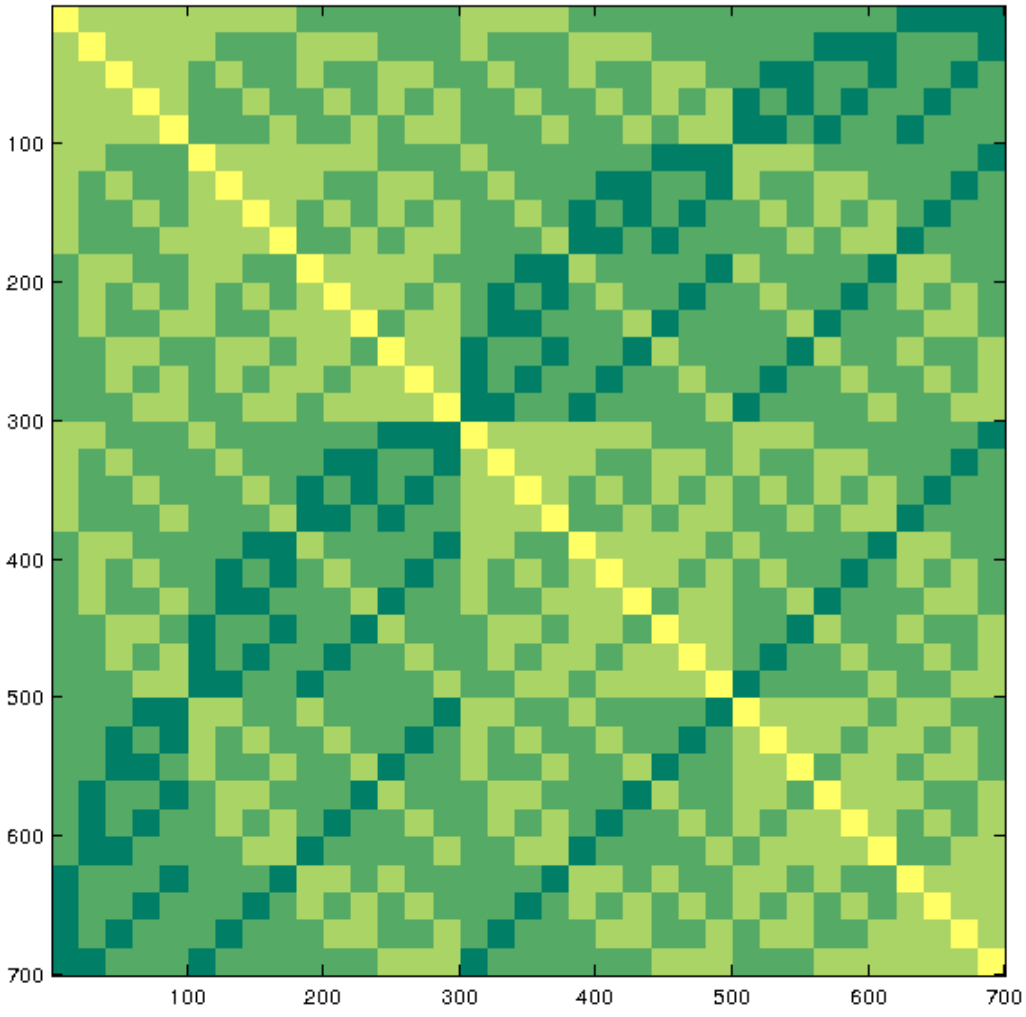
- If communities are very different sizes, random sampling will not find small communities.
- Instead sample ALL subsets of size $T = 100 \log(kd/\gamma)/\alpha_{\min}\epsilon^2$.
- Treat these subsets as the random subsamples in the previous algorithm.
- Since we have to check all subsets of size T, run time is $O(n^{T+C})$.

The Overlapping Community Model

- Assume that network has k communities and n users.
- Every user belongs to exactly d of them.
- The $\binom{k}{d}$ intersections of d communities are all nonempty, with cardinality at least $\hat{n}$.
- If u and v belong to ℓ of the same communities, they are attached by ℓ edges.
- Noise is introduced as up to m random edge additions and deletions between each pair of users.
- Rationale: probability of adding an edge between two users is proportional to the number of communities they share.

An example

- $k = 7, d = 3, \hat{n} = 20, p = 0.25, q = 0.25$



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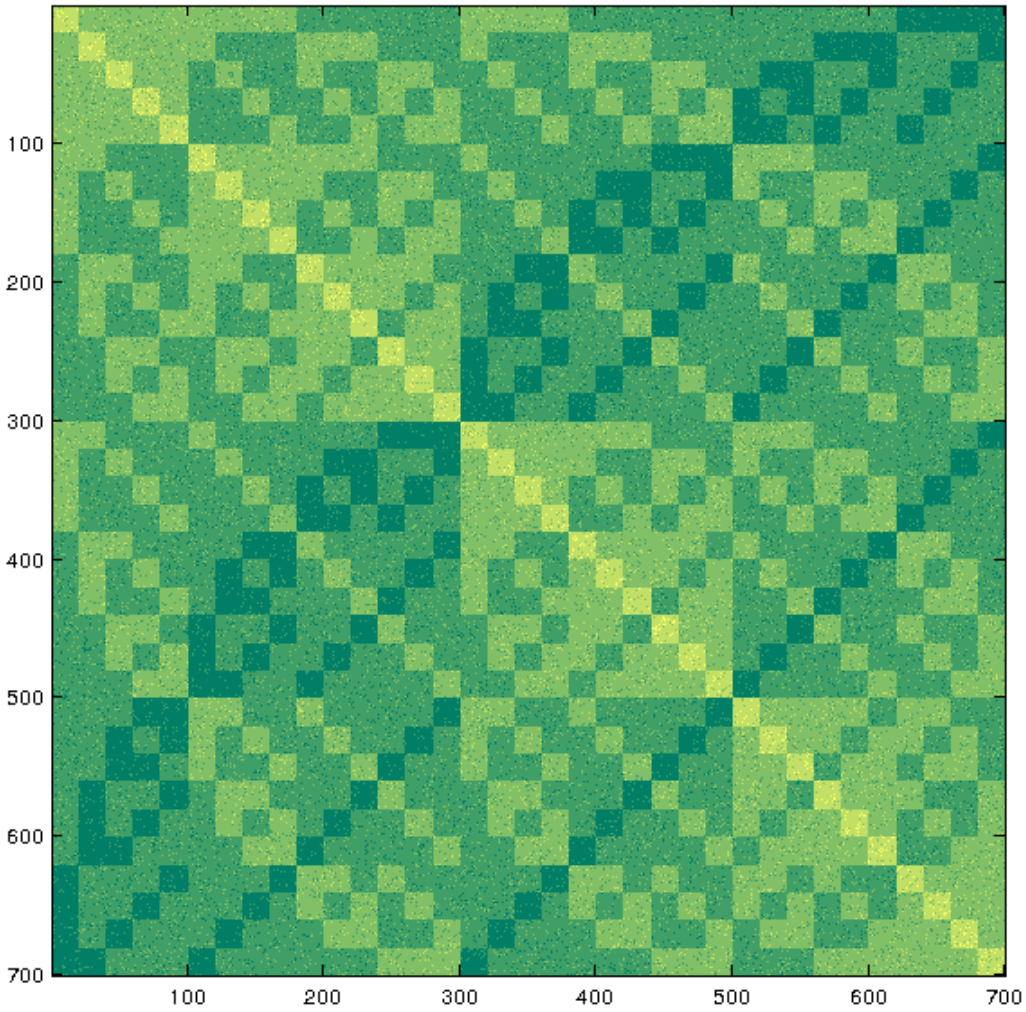

What if we knew the intersections of communities?

- The adjacency matrix of the multigraph is a $\binom{k}{d} \times \binom{k}{d}$ block matrix.
- Diagonal blocks correspond to the intersection d communities.
- These should be more denser than other blocks.
- The blocks are disjoint.
- Should be able to find by looking for $\binom{k}{d}$ clusters.

Guaranteed intersection recovery

Theorem

- *Suppose that the multigraph G is sampled from the overlapping community model.*
- *Then the set of intersections of d communities can be recovered w.h.p. from the optimal solution of the semidefinite program*

$$\max \left\{ Tr(AX) : X\mathbf{e} \leq \mathbf{e},\ TrX = \binom{k}{d},\ X \geq 0,\ X \in \Sigma_+^n \right\}$$

in the case that

$$\hat{n} = \Omega(\log n \sqrt{mn}).$$

Algorithm blue print: Contraction

Step (Initial clustering)

Find all intersections of d communities.

Step (Contraction)

- *Obtain the intersection graph G_I as follows.*
- *Contract each intersections of d communities to a single node.*
- *Each pair of nodes is adjacent by an edge with weight equal to the average number edges between the two intersections of communities.*
- *Edge weight is the number of shared communities in the two intersections w.h.p.*

Algorithm blue print: Union step

Step (Union of clusters)

- *Suppose that user u is in communities $S = \{1, 2, ..., d\}$.*
- *Delete any node in G_I with less than $d-1$ edges to $I_{\{1,2,...,d\}}$.*
  - *These nodes do not share $d-1$ communities with S w.h.p..*
  - *Cannot be in the intersection of $d-1$ communities containing u.*
- *Delete $I_{\{1,2,...,d\}}$ as well.*
- *Blocks of nodes sharing the same $d-1$ elements of S form d disjoint clusters in this reduced graph:*

$$I_{\{2,...,d\}}, I_{\{1,3,...,d\}}, \ldots, I_{\{1,2,...,d-1\}}.$$

- *Apply a clustering algorithm to find them.*

Algorithm blue print: Induction

Step

Repeat recursively

- *The union of $I_{\{1,\ldots,d\}}$ and each of the clusters found gives the set of intersections of $d-1$ communities containing u.*
- *Repeat the process to find the sets of intersections of $d-2, d-3, \ldots, 2$ communities containing u.*
- *Repeat one last time to find the k communities containing u.*

Complexity: Generate at most $d!$ graphs to cluster. If d is very small compared to n then the whole algorithm runs in polynomial time (in n, k).

How to cluster?

- How do we find the intermediate clusters/intersections at each step?
- Can our SDP find these clusters?
- Is there an inexpensive way to find these clusters?
- Do we need to solve an SDP? When would an LP suffice?
- Do we only have to compare degrees and common neighbours as in Arora et al.?

Are all branches necessary?

- The inductive step generates $O(d!)$ graphs to cluster (or SDPs to solve).
- Do we need to solve all of them to find all of the communities?
- How to choose which ones? Random sampling?

Different generative models?

- In the overlapping communities model, we assume that it is known how many communities a pair of users share.
- This is unlikely to be true in practice.
- What if we only required the intersections of communities to be more dense than communities and communities to be more dense than noncommunities?
- Can we recover the intersections of communities in this case?
- What if some intersections are empty?