Nuclear Norm Minimization for the Planted Clique and Biclique Problems

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Outline

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1. Nuclear Norm Minimization
2. Maximum Clique
   - The Adversarial Case
   - The Random Case

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Affine Rank Minimization

- **Affine rank minimization** refers to optimization problems of the form

$$\begin{array}{ll} \min & \operatorname{rank}(X) \\ \text{s.t.} & \mathcal{A}(X) = b \end{array}$$

  for linear $\mathcal{A} : \mathbf{R}^{m \times n} \rightarrow \mathbf{R}^p$, $b \in \mathbf{R}^p$.
- Known to be **NP-hard**.

Nuclear Norm Relaxation

- Solution suggested by **Recht-Fazel-Parrillo (2007)**.
- Replace $\text{rank}\,(X)$ with $\|X\|_* = \sigma_1(X) + \sigma_2(X) + \cdots + \sigma_{\min m,n}(X)$.
- **R-F-P** and **Candès-Recht (2008)** show that the relaxation

$$\begin{array}{ll} \min & \|X\|_* \\ \text{s.t.} & \mathcal{A}(X) = b \end{array}$$

returns the optimal solution to the original problem provided that $\mathcal{A}$ samples "enough" of the entries of X.

Compressive Sensing

- Rank minimization can be thought of as a generalization of **vector cardinality minimization**:

$$\begin{array}{ll} \min & \#\text{ nonzero entries of } x \\ \text{s.t.} & Ax = b \end{array}$$

  for $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$.
- Problem is NP-hard.
- Can be relaxed using the l_1-norm: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.
- For certain classes of random A, then the l_1-relaxation recovers exact solution to the original problem.

Properties of the Nuclear Norm

- $\|\cdot\|_*$ is convex. In fact, $\|\cdot\|_*$ is the **convex envelope** or **convex hull** of $\text{rank}(\cdot)$ on $\{X \in \mathbf{R}^{m\times n} : \|X\| \leq 1\}$.
- $\|\cdot\|_*$ can be calculated by solving the following SDP:

$$\|X\|_* = \begin{array}{ll} \max_{Y} & \langle X, Y\rangle \\ \text{s.t.} & \begin{bmatrix} I_m & Y \\ Y^T & I_n \end{bmatrix} \succeq 0 \end{array} \qquad (P)$$

$$= \begin{array}{ll} \min_{W_1,W_2} & \frac{1}{2}(\text{tr}(W_1) + \text{tr}(W_2)) \\ \text{s.t.} & \begin{bmatrix} W_1 & X \\ X^T & W_2 \end{bmatrix} \succeq 0. \end{array} \qquad (D)$$

Subdifferential of $\|\cdot\|_*$

Lemma (Watson (92))

- *Let X have rank r and SVD $X = \sum_{k=1}^{r} \sigma_k u_k v_k^T$.*
- *Then ϕ is a subgradient of $\|\cdot\|_*$ at X if and only if*

$$\phi = \sum_{k=1}^{r} u_k v_k^T + W$$

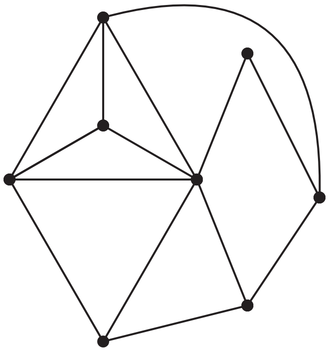
where

$$\|W\| \leq 1, \quad W^T u_k = 0 \; \forall \, k = 1, \ldots, r, \quad W v_k = 0 \;\; \forall \, k = 1, \ldots, r.$$

Maximum clique problem

- Given graph $G = (V, E)$, want to find the largest complete subgraph (clique) of G.
- One of the original NP-hard problems.
- Can be used as a simple model for data mining problems.

Example



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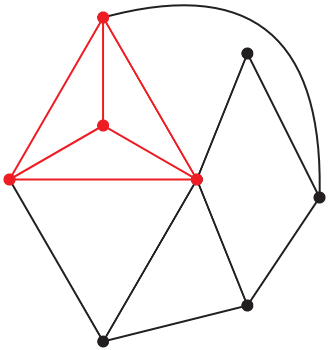


Formulation as affine rank minimization

- The adjacency matrix of the graph $H' = H \cup \{vv : v \in V(H)\}$ has rank equal to one if and only if H is a complete graph.
- Thus, a clique K of G containing n vertices can be found by solving

$$\begin{aligned} \min \quad & \operatorname{rank}(X) \\ \text{s.t.} \quad & \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n^2, \\ & X_{ij} = 0 \text{ if } (i,j) \notin E \text{ and } i \neq j, \\ & X \in [0,1]^{V \times V}. \end{aligned}$$

- Still NP-hard.

Nuclear norm relaxation of max clique

- We consider the convex relaxation

$$\begin{aligned} \min \quad & \|X\|_* \\ \text{s.t.} \quad & \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n^2, \\ & X_{ij} = 0 \text{ if } (i,j) \notin E \text{ and } i \neq j. \end{aligned}$$

- **Our result:** If G consists of a clique on n vertices plus some diversionary edges, then this convex relaxation recovers the maximum clique of G for sufficiently large n.

Optimality Conditions

Theorem

- *Let* $V^* \subset V$ *such that* $|V^*| = n$. *Let* $\bar{v}$ *be the characteristic vector of* V^* *Let* $X^* = \bar{v}\bar{v}^T$.
- *If there exists* $W \in \mathbf{R}^{V\times V}$ *and* $\lambda \in \mathbf{R}^{V\times V}$, $\mu \in \mathbf{R}_+$ *such that* $W\bar{v} = 0, \quad \bar{v}^T W = 0, \quad \|W\| \leq 1$ *and*

$$\frac{\bar{v}\bar{v}^T}{n} + W = \mu ee^T + \sum_{(i,j)\in\bar{E}} \lambda_{ij} e_i e_j^T .$$

then

  - X^* *is the optimal solution to the nuclear norm relaxation:*

$$\min\{\|X\|_* : \sum X_{ij} \geq n^2, \;\; X_{ij} = 0 \; \forall\, (i,j) \in \bar{E}\}.$$

  - *For every clique* K *of* G, $|V(K)| \leq n$.
  - *If* $\|W\| < 1$, $\mu > 0$, *the maximum clique spanned by* V^* *is unique.*

Proof

- Optimality of X^* follows immediately from the KKT conditions:

$$\partial \|X^*\|_* = \left\{ \frac{\bar{v}\bar{v}^T}{n} + W : Wv = 0, W^T v = 0, \|W\| \leq 1 \right\}.$$

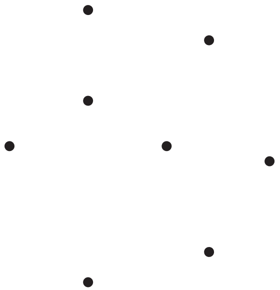
- Suppose V' spans a clique. Let $\bar{v}'$ be the characteristic vector of V'.
- Then $X' = \bar{v}'\bar{v}'^T (n^2/|V'|^2)$ is feasible for the relaxation.
- So

$$\|X'\|_* = \frac{n^2}{|V'|} \geq \|X^*\|_* = n.$$

- Therefore, $|V'| \leq n$.

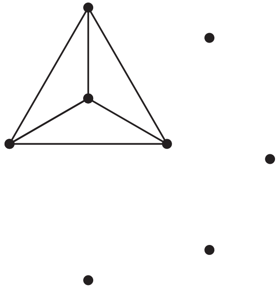
The Adversarial Case

- Suppose $G = (V, E)$ is generated as follows:



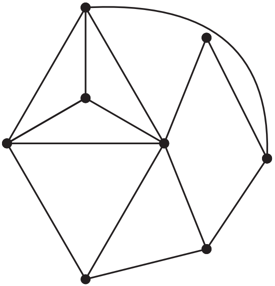
The Adversarial Case

- Suppose $G = (V, E)$ is generated as follows:
  - For $V^* \subset V$, for all $i, j \in V^*$ add (i, j) to E.



The Adversarial Case

- Suppose $G = (V, E)$ is generated as follows:
    - For $V^* \subset V$, for all $i, j \in V^*$ add (i, j) to E.
    - An adversary chooses a number of the remaining potential edges to be added to the graph.



Theorem

- *Suppose that every vertex in* $V - V^*$ *is adjacent to at most* δn *vertices in* V^* *for* $0 < \delta < 1$.
- *Then* G *can contain at most* $O(n^2)$ *edges other than those in* $V^* \times V^*$ *and* K_{V^*} *will still be the maximum clique of* G.

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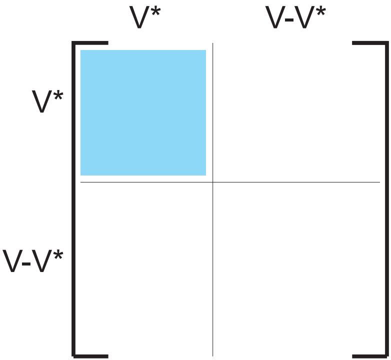
- To prove we want to find W, λ, μ satisfying the KKT conditions.
- Take $\mu = 1/n$.

Construction of W

(1) $(i,j) \in V^* \times V^*$:

$$W(i,j) = 0.$$



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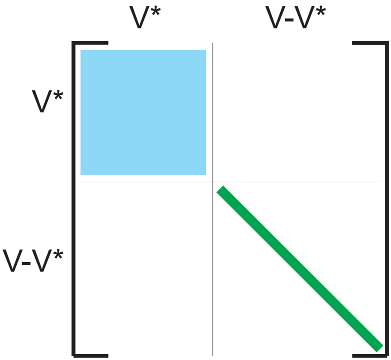

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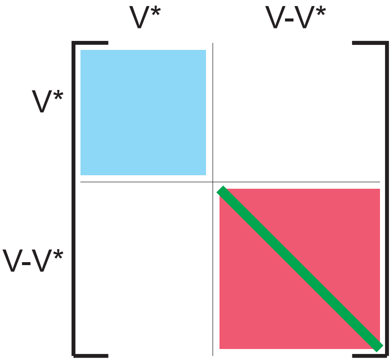

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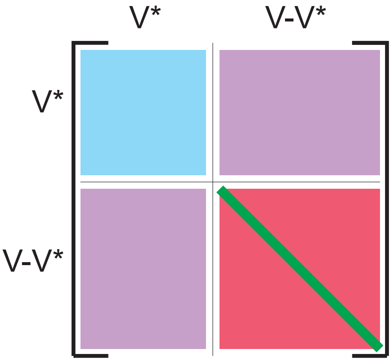
$$W(i,j) = 0.$$

(4) $(i,j) \in V^* \times (V - V^*)$, $ij \notin E$:

$$W(i,j) = -\frac{p_j}{n(n - p_j)};$$

p_j is the # of edges from j to V^*.



Proof: $W\bar{v} = 0$

- First show $W\bar{v} = 0$
- If $i \in V^*$, then $W(i,j) = 0$ for all $j \in V^*$. So

$$W(i,:)\bar{v} = \sum_{j \in V^*} W(i,j) = 0$$

- If $i \in V - V^*$,

$$W(i,:)\bar{v} = p_i \frac{1}{n} - (n - p_i)\frac{p_i}{n(n-p_i)} = 0.$$

- So $W\bar{v} = 0$.

Proof : $\|W\| \leq 1$

- Write $W = W_D + W_0$. Then

$$\|W\|^2 \leq 2(\|W_D\|^2 + \|W_0\|^2) \leq 2(1/n^2 + \|W_0\|^2).$$

- $\|W_0\|^2 \leq \|W_0\|_F^2 = \|W_0(V - V^*, V - V^*)\|_F^2 + 2\|W(V^*, V - V^*)\|_F^2$.
- At most r edges are not in the clique so

$$\|W_0(V - V^*, V - V^*)\|_F^2 \leq 2r/n^2.$$

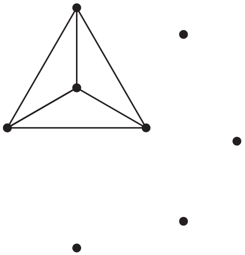
- Since $n - p_j \geq (1 - \delta)n$,

$$\|W(V^*, V - V^*)\|_F^2 = \sum_{j \in V - V^*} \frac{p_j}{n^2} + (n - p_j)\frac{p_j^2}{(n - p_j)^2}$$
$$\leq const \cdot \sum_{j \in V - V^*} \frac{p_j}{n^2} \leq const \cdot \frac{r}{n^2}.$$

- So $\|W\|^2 \leq 1$ if $r \leq const \cdot n^2$. □

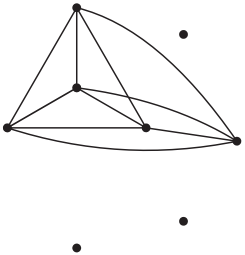
Example

- Suppose an adversary can add n edges from $v \in V - V^*$ to K_{V^*}:



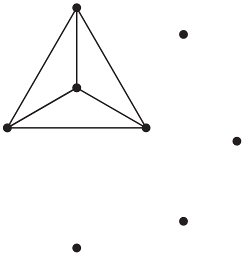
Example

- Suppose an adversary can add n edges from $v \in V - V^*$ to K_{V^*}:
- Then $K_{V^*} \cup \{v\}$ is a bigger clique.



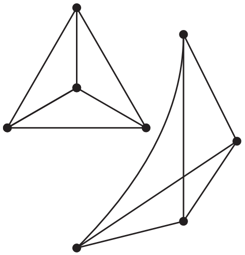
Example

- Suppose adversary can add n^2 edges not in the clique.



Example

- Suppose adversary can add n^2 edges not in the clique.
- Can form an additional clique of size n^2.



The Random Case

- Form $G = (V, E)$ as follows:
  - For $V^* \subset V$, for all $i, j \in V^*$ add (i, j) to E.
  - Add each remaining potential edge to the graph independently with probability p.

Feige-Krauthgamer Algorithm

- Suppose $G = (V, E)$ is randomly generated as on last slide with a clique of size $k > c\sqrt{|V|}$ for sufficiently large c.
- Then, with probability exponentially close to 1, $\vartheta(\overline{G}) = k$ and

$$\vartheta(\overline{G \setminus v}) = \begin{cases} k - 1 & \text{if } v \text{ is in the clique,} \\ k & \text{otherwise} \end{cases}$$

  for every $v \in V(G)$.
- **Algorithm:** Check every node v in G:
  - If $\vartheta(\overline{G \setminus v}) < \vartheta(\overline{G}) - 1/2$, add v to K.
  - When done, K is the maximum clique of G with high probability.

Theorem

- *Suppose* $n > \alpha\sqrt{|V|}$ *for sufficiently large* $\alpha > 0$.
- *Then* K_{V^*} *is the unique maximum clique of* G *and* X^* *is the optimal solution of the nuclear norm relaxation with probability exponentially close to 1.*

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- Take $\mu = 1/n$. Want to find W, λ that satisfy the optimality conditions with high probability.

Construction of W

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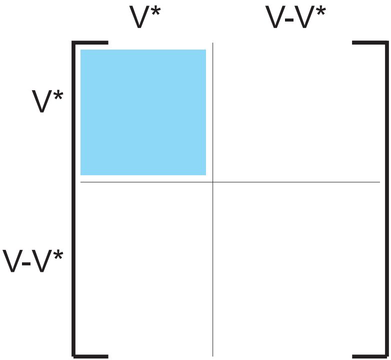

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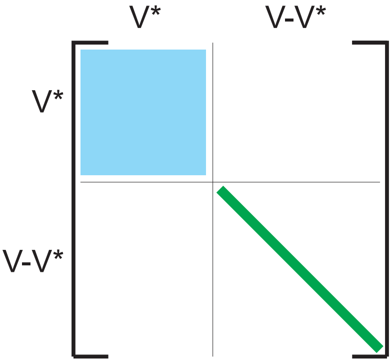

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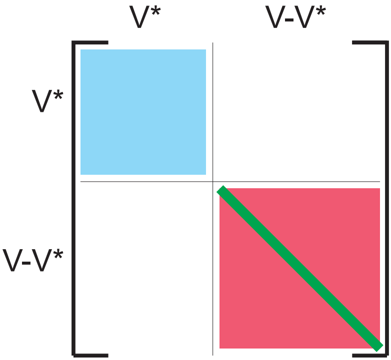
(3) $(i,j) \in (V - V^*) \times (V - V^*)$, $ij \notin E$:

$$W(i,j) = -\frac{p}{n(1-p)}.$$



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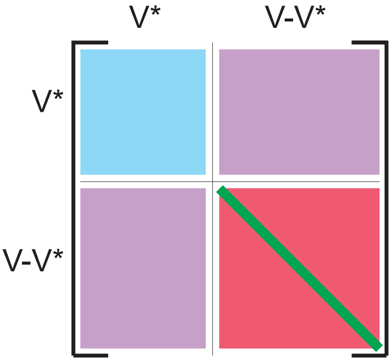

Proof

- Write $W = W_1 + W_2 + \cdots + W_5$.
- If $(i,j) \in V^* \times V^*$ or $i = j$ choose

$$W_1(i,j) = \begin{cases} 1/n & \text{with prob } p \\ -p/n(1-p) & \text{with prob } 1-p \end{cases}$$

- Otherwise,

$$W_1(i,j) = \begin{cases} 1/n & \text{if } (i,j) \in E \\ -p/n(1-p) & \text{otherwise} \end{cases}$$

Lemma (Füredi-Komlós)

- *Suppose a random symmetric matrix* $A = [A_{ij}] \in \Sigma^n$ *is defined as follows.*
  - *Take* $A_{ij} = 1$ *with probability* p, $-p/(1-p)$ *with probability* $1-p$ *for all* $1 \leq i \leq j \leq n$.
  - *Define* $A_{ji} = A_{ij}$ *for all* $j > i$.
- *Call this distribution* Ω.
- *Then*

$$\|A\| \leq 3\sigma\sqrt{n}$$

with probability $\geq 1 - \exp(-cn^{1/6})$ *for some* c *depending on* σ.

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with probability $\geq 1 - \exp(-cn^{1/6})$ *for some* c *depending on* σ.

- So $\|W_1\| \leq 3\sigma\sqrt{N}/n$ with probability at least $1 - \exp(c_1 N^{1/6})$

Proof (Cont'd)

- Let $W_2(i,j) = W(i,j) - W_1(i,j)$ for all $(i,j) \in V^* \times V^*$, $W_2(i,j) = 0$ otherwise.
- Applying the lemma again, $\|W_2\| \leq 3\sigma/\sqrt{n}$ with probability at least $1 - \exp(c_1 n^{1/6})$.

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- Let $W_3(i,i) = W(i,i) - W_1(i,i)$ for all $i \in V - V^*$ and $W_3(i,j) = 0$ otherwise.
- Then $W_3 \leq 2/n$.

Lemma

- *Suppose* $A \in \mathbf{R}^{M \times N}$ *with entries chosen according to* Ω.
- *Define* $\tilde{A} \in \mathbf{R}^{M \times N}$ *by*

$$\tilde{A}_{ij} = \begin{cases} 1 & \text{if } A_{ij} = 1, \\ -n_j/(n - n_j) & \text{if } A_{ij} = -p/(1-p) \end{cases}$$

where n_j *is the # of 1's in column* j *of* A.

- *Then there exists* $c_1 > 0, 0 < c_2 < 1$ *depending on* p *such that*

$$\|A - \tilde{A}\|_F^2 \leq c_1 N \quad \text{and} \quad n_j < M \;\; \forall j = 1, 2, \ldots, N$$

with probability at least $1 - (2/3)^N - N c_2^M$.

Proof (Conclusion)

- Take $W_4(V^*, V - V^*) = W(V^*, V - V^*) - W_1(V^*, V - V^*)$, W_4 zero in all other entries.
- So $W_4(V^*, V^*)$ is in the form $A - \tilde{A}/n$ from the lemma:

$$\|W_4\|^2 \leq \|W_4\|_F^2 = \|W_4(V - V^*)\|_F^2 \leq \alpha N/n^2$$

  for some $\alpha > 0$ with probability tending exponentially to 1 as $N \to \infty$ if $n = \Omega(N)$.
- Let $W_5 := W_4^T$. Then

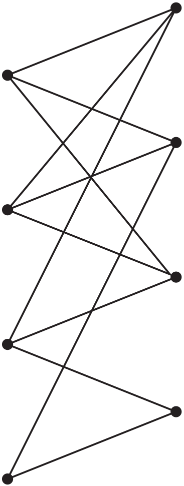
$$\|W_5\|^2 \leq \alpha N/n^2$$

  with the same probability.

The Maximum Edge Biclique Problem

- Given a bipartite graph $G = (U, V, E)$, the **maximum edge biclique problem** is to find a complete bipartite subgraph of G with maximum number of edges.
- **Peeters:** MEBP is NP-hard.

Example



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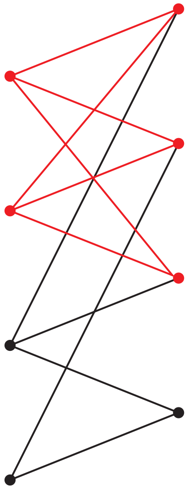


Nuclear norm relaxation of MEBP

- A biclique G containing mn edges can be found by solving

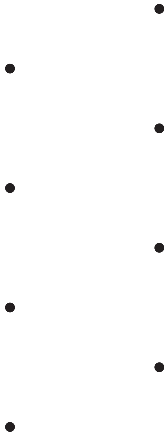
$$\begin{aligned} \min \quad & \operatorname{rank}(X) \\ \text{s.t.} \quad & \sum_{i \in U} \sum_{j \in V} X_{ij} \geq mn, \\ & X_{ij} = 0 \text{ if } (i,j) \notin E \\ & X \in [0,1]^{U \times V}. \end{aligned}$$

- We consider the convex relaxation

$$\begin{aligned} \min \quad & \|X\|_* \\ \text{s.t.} \quad & \sum_{i \in U} \sum_{j \in V} X_{ij} \geq mn, \\ & X_{ij} = 0 \text{ if } (i,j) \notin E. \end{aligned}$$

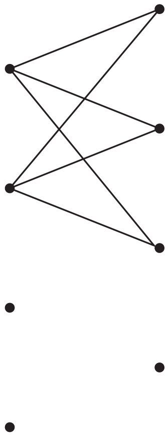
The Adversarial Case

- Suppose bipartite $G = (U, V, E)$ is generated as follows:



The Adversarial Case

- Suppose bipartite $G = (U, V, E)$ is generated as follows:
  - First add a biclique $U^* \times V^*$ with $U^* \subseteq U$, $V^* \subseteq V$. Let $|U^*| = m$, $|V^*| = n$.



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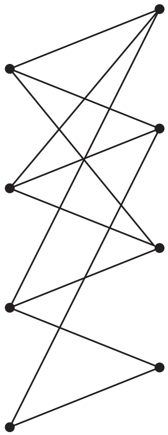
  - First add a biclique $U^* \times V^*$ with $U^* \subseteq U$, $V^* \subseteq V$. Let $|U^*| = m$, $|V^*| = n$.
  - An adversary chooses a number of the remaining potential edges to be added to the graph.



Theorem

Suppose that

- *G contains at most r edges not in* $U^* \times V^*$,
- *each vertex of* $V - V^*$ *is adjacent to at most* αm *vertices of* U^* *for* $\alpha \in (0, 1)$, *and*
- *each vertex of* $U - U^*$ *is adjacent to at most* βn *vertices of* V^* *for* $\beta \in (0, 1)$.

Then there exists constant $c \in (0, 1)$ *depending on* α, β *such that* $U^* \times V^*$ *is the unique maximum edge biclique of G if* $r < cmn$.

- As with the maximum clique problem, this bound is the best possible.

The Random Case

- Suppose bipartite $G = (U, V, E)$ such that $|V| = N$, $|U| = \lceil yN \rceil$ for some $y > 0$ is generated as follows:
  - First add biclique $U^* \times V^*$ with $|V^*| = n$, $|U^*| = \lceil zn \rceil$ for some $z > 0$
  - Add each remaining potential edge to the graph with probability p (independently).

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Theorem

If $n = \Omega(\sqrt{N})$ then $U^* \times V^*$ is the unique maximum edge biclique of G with probability tending exponentially to 1 as $N \to \infty$ and $X^* = \bar{u}\bar{v}^T$ is the unique solution to the nuclear norm relaxation.

Conclusions and Future Work

- Have shown that for two hard combinatorial problems, nuclear norm minimization returns the exact solution in poly-time in certain cases.
- Future work:
  - Apply the same technique to other hard problems (NMF, clustering).
  - Specialized algorithms.